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VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C8T

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) State and prove the necessary and sufficient condition for integrability of a bounded function.

(b) Show by an example that if $|f(x)|$ is integrable then $f(x)$ may not be integrable. 8+7

2. (a) Prove that every continuous function is integrable.

(b) Applying Second Mean Value Theorem of Bonnet's form show that

$$\left|\int_{x'}^{x''} \frac{sinx}{x}dx\right| \le \frac{2}{x'} \quad \text{where} \quad 0 \le x' \le x'' .$$

(c) Show for $k^2 < 1$, $\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \times \frac{1}{\sqrt{1-\frac{1}{4}\times k^2}}$. 5+4+6

3. (a) Verify Second Mean Value Theorem of Weierstrass form for the function $x^2\, cosx$ in the Interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) Show that $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$.

(c) State and prove the fundamental theorem of integral calculus.

5+3+7

4. (a) Show that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

(b) Discuss the convergence of $\int_0^\infty e^{-x}x^{n-1}dx$.

(c) Show that $\int_0^\infty \frac{cosx}{\sqrt{1+x^3}}dx$ converges absolutely but $\int_0^\infty \frac{cosx}{\sqrt{1+2x^2}}dx$ diverges.

5+5+5

5. (a) Examine the pointwise convergence of the sequence of function $\{f_n\}$ on R defined by $f_n(x) = x^n$.

(b) If $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous and $f_n(x)=f\left(x+\frac{1}{n}\right)$. Then prove that $\{f_n\}$ converges uniformly to f on $\mathbb{R}$.

(c) Let for each n≥2, $f_n(x)=\begin{cases} n & if \quad 0<n<\frac{1}{2} \\ 0 & otherwise \end{cases}$

Obtain $\lim_{n\to\infty} f_n(x)$ in [0,1] and verify that $\lim_{n\to\infty}\int_0^\infty f_n(x) \neq \int_0^1 lim_{n\to\infty} f_n(x)\,dx$.

4+4+7

6. (a) Check the uniform convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n^2\left(1+x^{2n}\right)}, x\in\mathbb{R}$$

(b) Let $f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}, x\in[0,1]$.

Show that $\int_0^1\left(\sum_{n=1}^{\infty}f_n(x)\right)dx=\sum_{n=1}^{\infty}\left(\int_0^1 f_n(x)\,dx\right)$; although the series $\sum_{n=1}^{\infty}f_n(x)$ is not uniformly convergent on [0,1].

(c) For the series $\sum_{n=1}^{\infty}f_n(x)$, where $f_n(x)=n^2xe^{-n^2x^2}-(n-1)^2xe^{-(n-1)^2x^2}, x\in[0,1]$,

show that $\sum_{n=1}^{\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1\left(\sum_{n=1}^{\infty}f_n(x)\right)dx$.

Is the series $\sum_{n=1}^{\infty} f_n(x)$ uniformly convergent on $[0,1]$? 3+6+6

7. (a) Expand *f(x)* = x in Fourier series in the interval $-\pi \le x \le \pi$.

(b) Prove that the even function *f(x)* = $|x|$ in $-\pi \le x \le \pi$ has a cosine series in Fourier's form.

Use Dirichlet's conditions of convergence to show that the series converges to $|x|$

throughout $-\pi \le x \le \pi$. 7+8

8. (a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ and discuss its convergence at each end of the interval.

(b) Find the series for *log (1+x)* by integration and hence use Abel's theorem to show that

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$. (3+2)+(7+3)

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VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 8 - T

Riemann Integration and Series of Functions

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

1. Answer any ***five*** questions : 2×5=10

(a) Let $f:[a,b]\to R$ be a bounded function and P be any partition over [a, b]. Define lower sum $L(P,f)$ and upper sum $U(P,f)$.

(b) Let $f:[a,b]\to R$ be integrable on [a, b]. If M and m be respectively the supremum and infimum of f on [a, b], prove that $m(b-a)\le\int_a^b f dx\le M(b-a)$.

(c) Prove or disprove : if f is differentiable on [0, 1], the relation $\int_0^1 f' dx = f(1)-f(0)$ is not always true.

P.T.O.

(d) A function f is continuous in the interval $[a, \infty)$ and $f(x) \to A(\neq 0)$ as $x \to \infty$.

Can the integral $\int_a^\infty f(x)\,dx$ converge?

(e) Discuss the convergence of $\int_0^1 e^{-x}.x^{n-1}dx$.

(f) Give examples of (i) everywhere convergent power series (ii) nowhere convergent power series.

(g) Let D be a finite subset of R. If a sequence of real valued functions $\{f_n(x)\}_n$ on D converges pointwise to $f(x)$, then show that it also converges uniformly to $f(x)$.

(h) Let $\sum_n f_n(x)$ be a series of functions defined on $D(\subset R)$. Explain when this series is said to be uniformly convergent on D.

2. Answer any ***four*** questions : 5×4=20

(a) Find the Fourier series of the periodic function f with period 2π, where

$f(x) = \begin{cases} 0, -\pi < x < a \\ 1, \ \ a \le x \le b \\ 0, \ \ b < x < \pi \end{cases}$. Find the sum of the series at $x = 4\pi + a$ and deduce that

$$\sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi-(b-a)}{2}.$$

(b) Evaluate $\int_2^5 (x^2 - x)\,dx$ by using the geometric partition of [2, 5] into n subintervals.

(c) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1}.\frac{x^n}{n}$ and discuss its convergence at each end of the interval.

(d) Show that $\sum_{n=0}^{\infty} x^n$ uniformly on $[-a, a]$ where $0 < a < 1$, but

$\sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2 x^2}\right]$ is not uniformly convergent on R.

P.T.O.

(e) Show that $\int_1^\infty x^{m-1}(\log x)^n dx$ is convergent if and only if $m<0,\ n>-1$.

(f) Let f be a continuous function on R and define $F(x)=\int_{x-1}^{x+1} f(t)\,dt,\ x\in R$. Show that F is differentiable on R and compute F'.

3. Answer any ***three*** questions : 10×3=30

(a) (i) State and prove the fundamental theorem of integral calculus.

(ii) If $0\le x\le 1$ then show that $\frac{x^2}{\sqrt{2}}\le\frac{x^2}{\sqrt{1+x}}\le x^2$ and hence show that $\frac{1}{3\sqrt{2}}\le\int_0^1\frac{x^2}{\sqrt{1+x}}\le\frac{1}{3}$. 5+5

(b) (i) If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is R —integrable over $[a, b]$. 3+3

(ii) Show that $\int_\pi^\infty \frac{\sin x}{x}dx$ converges but not absolutely. 4

(c) (i) Let $\sum_n u_n(x)$ be a series of real valued function defined on $[a, b]$ and each $u_n(x)$ is R—integrable on $[a, b]$. If the series converges uniformly to f on $[a, b]$, then prove that f is R —integrable on $[a, b]$ and

$$\int_a^b\left[\sum_{n=1}^\infty u_n(x)\right]dx=\sum_{n=1}^\infty\int_a^b u_n(x)\,dx.$$

Give an example to show that the condition of uniforms convergence of $\sum_n u_n(x)$ is only a sufficient condition but not necessary. 5+2

(ii) Find the region of convergence of the series $\sum_{n=1}^\infty\frac{x^{3n}}{2^n}$. 3

(d) (i) Verify that the function $y = x^3 \sin\frac{1}{x}$ for $x \neq 0$ and $y = 0$ for $x = 0$ in the interval $[-\pi, \pi]$ is continuous together with its first derivative but does not satisfy the conditions of Dirchlet's theorem. Can it be expanded into a Fourier series in the interval $[-\pi, \pi]$. 5

(ii) Prove that the integral $\int_0^{\frac{\pi}{2}} \sin x \log \sin x \, dx$ exists and find its value. 5

(e) (i) Let $f_n(x) = |x|^{1+\frac{1}{n}}, x \in [-1, 1]$. Show that $\{f_n\}_n$ is uniformly convergent on [–1, 1]. Also show that each f_n is differentiable on [–1, 1] but the limit function is not differentiable for all x in [–1, 1]. 2+2+2

(ii) Prove or disprove : $\{\tan^{-1} nx\}_n$ is not uniformly convergent on any interval which includes zero. 4

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VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C9T

MULTIVARIATE CALCULUS

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) Let $f(x,y)=\begin{cases} x\sin\frac{1}{y}+y\sin\frac{1}{x}, xy\neq 0; \\ 0, xy=0. \end{cases}$

Show that at (0,0) the double limit exists but the repeated limits do not exist.

(b) Let $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, x^2+y^2\neq 0;\\ 0, x^2+y^2=0.\end{cases}$

Prove that f is a continuous function of either variable when the other variable is given a fixed value. Is f continuous at (0, 0)? Justify.

(c) If $u = f(x,y)$, where $x=r\cos\theta, y=r\sin\theta$; prove that

(i) $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$;

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$. 4+4+7

2. (a) When is a function $f(x,y)$ said to be differentiable at a point (x,y)? State the sufficient condition for differentiability of (x,y).
Verify the sufficient condition for differentiability of the following function

$$f(x,y)=\begin{cases}x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}, x\neq 0, y\neq 0;\\ x^2\sin\dfrac{1}{x}, x\neq 0, y=0;\\ y^2\sin\dfrac{1}{y}, x=0, y\neq 0;\\ 0, x=0, y=0.\end{cases}$$

(b) Let $f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x-y}, x\neq y;\\ 0, x=y\ .\end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0,0). Examine the continuity of *f(x,y)* at (0, 0).

(c) If H be a homogeneous function in x and y of degree n having continuous first order partial derivatives and $u(x,y)=\left(x^2+y^2\right)^{-n/2}$, show that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)=0$. 6+4+5

3. (a) Let $f(x,y)=\begin{cases}\frac{x^2 y}{x^4+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0).\end{cases}$

Show that f has a directional derivative at (0,0) in any direction $\beta=(l,m),\ l^2+m^2=1$, but f is discontinuous at (0, 0).

(b) If a function *f(x,y)* defined in a certain domain *D* of the xy-plane where $(a,b)\in D$ be such that both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in some neighbourhood of (a,b) and both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable at *(a,b)*, then prove that $f_{xy}(a,b) = f_{yx}(a,b)$.

(c) For the function $f(x,y)=\begin{cases}\frac{x^2 y^2}{x^2+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0);\end{cases}$

show that $f_{xy}(0,0)=f_{yx}(0,0)$. 4+6+5

4. (a) Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *is* $\frac{8abc}{3\sqrt{3}}$.

(b) Find the stationary points of $f(x,y,z) = x^2y^2z^2$ subject to the condition $x^2 + y^2 + z^2 = a^2$ (x, y, z are positive).

(c) Show that $\iiint (x+y+z)x^2y^2z^2 dx\ dy\ dz = \frac{1}{50400}$ taken throughout the tetrahedron bounded by three coordinate planes and x + y + z = 1.

5+4+6

5. (a) If E be the region bounded by the circle $x^2 + y^2 - 2ax - 2by = 0$, show that

$$\iint_E \sqrt{x(2a-x)+y(2b-y)}\,dx\ dy = \frac{2\pi}{3}\left(a^2+b^2\right)^{\frac{3}{2}}.$$

(b) Prove that $\iiint_V \frac{dx\ dy\ dz}{x^2 + y^2 + \left(z - \frac{1}{2}\right)^2} = \pi\left(2 + \frac{3}{2}\log 3\right)$

where V = $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$.

7+8

6. (a) In which direction from the point (1,3,2), the directional derivative of $\phi = 2xz - y^2$ is maximum? What is the magnitude of this maximum?

(b) Is there a differentiable vector function $\vec{v}$ such that $curl\ \vec{v} = \vec{r}$? Justify it. Show that $\vec{E} = \frac{\vec{r}}{r^2}$ is irrotational. Find ϕ such that $\vec{E} = -\vec{\nabla}\phi$ and such that $\phi(a) = 0$ where $a > 0$.

(c) If $\vec{A} = \left(4xy - 3x^2z^2\right)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$, prove that $\int_C \vec{A} \bullet d\vec{r}$ is independent of the curve C joining two given points.

Is $\vec{A} \bullet d\vec{r}$ an exact differential? If yes, then solve the differential equation $\vec{A} \bullet d\vec{r} = 0$.

3+6+6

7. (a) Prove $\nabla^2 r^n = n(n+1)r^{n-2}$, where n is a constant, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(b) Prove that if $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \vec{F} \cdot d\vec{r} = 0$ for all closed paths in the region and conversely.

(c) Verify Green's theorem in the plane for $\oint_C \left(3x^2 - 8y^2\right)dx + (4y - 6xy)\,dy$, where C is the boundary of the region enclosed by : $y = \sqrt{x}, y = x^2$ 5+4+6

8. (a) Show that $\vec{F} = \left(2xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Also evaluate the work done in moving an object in this field from (1,-2,1) to (3,1,4).

(b) Prove $\iint_S r^5 \hat{n}\, dS = \iiint_V 5r^3 \vec{r}\, dV$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(c) Evaluate by Stokes' theorem $\oint_C \sin z\, dx - \cos x\, dy + \sin y\, dz$, where C is the boundary of the rectangle : $0 \le x \le \pi, 0 \le y \le 1, z = 3$. 6+5+4

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VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 9 - T

Multivariate Calculus

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

1. Answer any ***five*** questions : 2×5=10

(a) Distinguish between double integral and repeated integral.

(b) Show that $\vec{\nabla}|\vec{r}|^n = n|\vec{r}|^{n-2}\,\vec{r}$, where $\vec{r} = xi + yj + zk$.

(c) Show that $\vec{\nabla}\Phi$ is a vector perpendicular to the surface $\Phi(x, y, z) = c$, where c is a constant.

(d) Write down the formula for the evaluation of length of a curve. Justify it.

(e) Show that $\lim\limits_{(x,y)\to 0} \dfrac{x - y^2}{x^2 + y^2}$ does not exist.

P.T.O.

(f) Find the equation of the tangent plane to the surface $f(x,y) = x^2 + y^2 + \sin xy$ at the point (0, 2, 4).

(g) Find the surface area of a sphere by using surface of revolution.

(h) If $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A} \times \vec{B}$ is irrotational.

2. Answer any ***four*** questions : 5×4=20

(a) State and prove the Schwartz's theorem for the equality of f_{xy} and f_{yx} at some point (a, b) of the domain of definition of $f(x, y)$.

(b) Express $\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x^2\, dy$ as a double integral and evaluate it.

(c) Prove $\vec{\nabla} \times (\vec{F} \times \vec{G}) = \vec{F}(\vec{\nabla}.\vec{G}) - \vec{F}.\vec{\nabla}\vec{G} + \vec{G}.\vec{\nabla}\vec{F} - \vec{G}(\vec{\nabla}.\vec{F})$, where $\vec{F}$ and $\vec{G}$ are differentiable vector function.

(d) Find $\iint_R f(x,y)\,dxdy$, over the region R bounded by $x = y^{\frac{1}{3}}$ and $x = \sqrt{y}$ where $f(x,y) = x^4 + y^2$.

(e) What is the maximum directional directional derivative of $g(x,y) = y^2 e^{2x}$ at (2, –1) and in the direction of what unit vector does it occur?

(f) Let f and g be twice differentiable functions of one variable and let $u(x,t) = f(x+ct) + g(x-ct)$ for a constant c. Show that $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

3. Answer any ***three*** questions : 10×3=30

(a) (i) Find the minimum value of $x^2 + y^2 + z^2$ subject to the constaint $ax + by + cz = 1(a \neq 0, b \neq 0, c \neq 0)$.

(ii) Show that $f(x,y,z) = (x^2, y^2, z^2)^{-\frac{1}{2}}$ is harmonic. 8+2

(b) (i) Let z be a differentiable function of x and y and let $x = r\cos\theta, y = r\sin\theta$,

Prove that $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$. 7

(ii) Prove that $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$ is not continuous at (0, 0). 3

(c) (i) Prove that $\iiint \frac{dxdydz}{x^2+y^2+(z-2)^2} = \pi\left(2 - \frac{3}{2}\log 3\right)$, extended over the sphere $x^2+y^2+z^2 \leq 1$.

(ii) Using a double integral, prove that the relation $B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$,

m, n > 0. 5+5

(d) (i) Verify Stoke's theorem for the function $\vec{F} = x^2 i - xyj$ integrated round the square in the plane z = 0 and bounded by the lines x = 0, y = 0, $x = a$, $y = a$.

(ii) Prove that $\iint\left[2a^2 - 2a(x+y) - (x^2+y^2)\right] dxdy = 8\pi a^4$, the region of integration being the interior of the circle $x^2+y^2+2a(x+y) = 2a^2$. 6+4

(e) (i) Evaluate $\iint_S \overline{A} . \hat{n}\, ds$; $\overline{A} = 2yi - zj + x^2 k$ over the surface S of the bounded by the parabolic cylinder $y^2 = 8x$, in the first octant bounded by the plane y = 4 and z = 6. 7

(ii) Find the directional derivative of $f(x,y) = 2x^2 - xy + 5$ at (1, 1) in the direction of unit vector $\left(\frac{3}{5}, -\frac{4}{5}\right)$. 3

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VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C10T

RING THEORY & LINEAR ALGEBRA – I

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) Let S be the set of all twice differentiable real-valued functions on $\mathbb{R}$ having second derivative zero at $x = 0$. Does $(S, +, \cdot)$ form a ring, where $(f+g)(x) = f(x) + g(x)$ and $(f\cdot g)(x) = f(x)g(x)$ for all $x \in \mathbb{R}$ and $f, g \in S$?

(b) Give an example (with reason) of a left ideal of a ring which is not a right ideal.

(c) Find all homomorphisms from the ring $\mathbb{Z}$ onto itself. Prove that $(\mathbb{Z}, +, \cdot)$ is not isomorphic with $(2\mathbb{Z}, +, \cdot)$ as rings.

(d) Let S be the set of all 2×2 real skew-symmetric matrices over $\mathbb{R}$. Prove that S forms a vector space over R and hence find a basis of S.

(e) Find a linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ so that ker $T = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, x - y - z = 0\right\}$.

2+2+(2+2)+(2+2)+3

2. (a) Let R be a ring, $x \in R$ and $\langle x \rangle$ denote the smallest ideal of R containing x. Prove that

$$\langle x \rangle = \{rx + xs + \sum_{i=1}^{m} s_i x t_i + nx \mid r, s, s_i, t_i \in R;\ m \in \mathbb{N},\ n \in \mathbb{Z}\}$$

(b) Let I and J be two ideals of a ring R. Show that $R/(I \cap J)$ is isomorphic to a subring of $R/I \times R/J$.

(c) Consider the set of vectors $B = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_k\}$ in a vector space V over the field F. Prove that B is linearly dependent if and only if at least one of the vectors of the set B can be expressed as a linear combination of the remaining vectors of B.

(d) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator that maps the basis vectors α, β, γ to $\alpha + \beta$, $\beta + \gamma$, γ, respectively. Show that T is an isomorphism.

5+3+4+3

3. (a) In a commutative ring with identity, prove that every maximal ideal is prime. If the ring lacks the identity, then show that the above result may not be true.

(b) Define characteristic of a ring. Give an example of a finite ring whose characteristic is 4.

(c) Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the subspace of all matrices of the form $\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$ and W_2 be the subspace of all matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

(d) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$T(a, b, c) = (3a,\ a - b,\ 2a + b + c)$.

Show that T is invertible. For any $(x,y,z)\in\mathbb{R}^3, T^{-1}(x,y,z)$?

(3+1)+(1+2)+4+(2+2)

4. (a) Does there exist an integral domain with 15 elements? Give justification in support of your answer.

(b) Find all ring homomorphism from $(\mathbb{Z}_{12},+,\cdot)$ to $(\mathbb{Z}_{30},+,\cdot)$.

(c) Let us consider the vector space

$$V = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} | \, a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}, a_{11} + a_{22} = 0\}$$

over the field of real numbers with the usual operations of matrix addition and multiplication of a matrix by a scalar. Find a basis for V and find *dim* V.

(d) Let V be the vector space of all polynomials over $\mathbb{R}$ with degree three or less. Consider the differentiation operator D on V. Then compute the following :

(i) First find the matrix representation of D with respect to the ordered basis $B = \{1, x, x^2, x^3\}$ of V.

(ii) Then consider another basis $B_1 = \{g_1, g_2, g_3, g_4\}$ of V where $g_i(x) = (x+t)^{i-1}$ for all i=1, 2, 3, 4 and for some $t\in\mathbb{R}$. Find the change of basis matrix with respect to B and B_1.

(iii) Hence find the matrix representation of D with respect to the ordered basis B_1 using the change of basis matrix and the matrix representation of D with respect to the ordered basis B.

3+3+3+(2+2+2)

5. (a) Give example of (i) a ring R and a subring S of R such that 1_R, 1_S both exist but $1_R \neq 1_S$; (ii) a ring R and a subring S of R such that 1_S exists but R does not contain the multiplicative identity.

(b) Let R and S be two rings and $f: R \to S$ be an epimorphism. Show that if I is an ideal of R then $f(I)$ is an ideal of S. Give an example to show that the result is not true if f is a homomorphism which is not surjective.

(c) Find four vectors $\alpha, \beta, \gamma, \delta \in \mathbb{R}^4$ such that $\{\alpha, \beta, \gamma, \delta\}$ is a linearly dependent subset but any three of them are linearly independent.

(d) Let V be an n dimensional vector space over the field F and W be an m dimensional vector space over the field F. Then prove that the dimension of the vector space $L(V, W) = \{T: V \to W \mid T \textit{ is a linear transformation}\}$ is mn by constructing a basis of $L(V, W)$.

(2+2)+(2+2)+3+4

6. (a) Let R be a commutative ring with identity $1_R \neq 0$. Prove that an ideal M of R is maximal if and only if the quotient ring R/M becomes a field.

(b) Suppose R is an integral domain and there is a ring homomorphism from $\mathbb{Z}$ onto the integral domain R. Then show that either $R \cong \mathbb{Z}$ or $R \cong \mathbb{Z}_p$ for some prime p.

(c) Extend the set$A$ = $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of the vector space $\mathbb{R}^4$ over $\mathbb{R}$.

(d) Prove that any n dimensional vector space over the field F is isomorphic with the space F^n over F.

4+3+4+4

7. (a) Find the field of quotients of the integral domain $\mathbb{Z}[i]$.

(b) For a commutative ring R with identity, do R and $R[x]$ always have the same characteristic ? Justify your answer.

(c) Let V be a vector space over the field F and W be a subspace of V. Show that

$\dim(V/W) = \dim V - \dim W$.

(d) Give examples of two linear operators U, T on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ such that $TU = 0$ but $UT \neq 0$ where 0 denotes the zero operator.

4+3+5+3

8. (a) Let R be an integral domain with finite number of ideals. Show that R is a field. Hence conclude that finite integral domain is a field.

(b) Consider a ring homomorphism $f: R \to S$ where R, S both are commutative rings with identity. If J is a prime ideal of S then is $f^{-1}(J)$ a prime ideal in R? Justify your answer.

(c) Let V be the vector space $\mathbb{R}^4$ over the $\mathbb{R}$ and W be the subspace of V generated by $\{(1, 0, 0, 0), (1, 1, 0, 0)\}$. Find a basis of the quotient space V/W.

(d) Prove that for any two linear operators $T: \mathbb{R}^3 \to \mathbb{R}^2$, $S: \mathbb{R}^2 \to \mathbb{R}^3$, ST is neither one-one nor onto.

(e) Let U, V, W be three finite dimensional vector spaces over a field F. Let $T: V \to W$ be a linear transformation and $S: W \to U$ be an isomorphism. Prove that

$\dim \ker(T) = \dim \ker(ST)$.

(3+2)+3+3+2+2

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VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 10 - T

Ring Theory and Linear Algebra - I

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

1. Answer any ***five*** questions : $2\times5=10$

   (a) Let X be any set and R be the power set of X. Does $(R, +, \cdot)$ form a ring where $A+B = A\cup B$ and $A\cdot B = A\cap B$ for all $A, B\in R$.

   (b) Extend $S = \{(1,1,0), (1,1,1)\}$ to a basis of the vector space $\mathbb{R}^3$ over $\mathbb{R}$.

   (c) Find the total number of units in the ring $M_2(Z_3)$, with usual notations.

   (d) Let V be a vector space over the field F and W_1, W_2 be two subspaces of V. Is $W_1\cup W_2$ a subspace of V?

   (e) In the ring $M_2(\mathbb{Z})$ of all 2×2 matrices over $\mathbb{Z}$, check whether the set $\left\{\begin{pmatrix}0 & a\\ 0 & b\end{pmatrix} \mid a,b\in\mathbb{Z}\right\}$ forms an ideal or not.

P.T.O.

(f) Determine the linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ over $\mathbb{R}$ such that T sends the vectors (1,0,0), (0,1,0), (0,0,1) to (0,1,0), (0,0,1), (1,0,0), respectively.

(g) Show that the mapping $f:\mathbb{Z}\sqrt{2} \to M_2(\mathbb{R})$ (where $M_2(\mathbb{R})$ denotes the ring of all 2 × 2 real matrices) defined by $f\left(a+b\sqrt{2}\right)=\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ is a homomorphism of rings.

(h) Give an example of a linear operator T on a vector space V such that ker T = *Im T*.

2. Answer any ***four*** questions : 5×4=20

(a) Let I denote the set of all polynomials in $\mathbb{Z}[x]$ with constant term of the form $4k\,(k \in \mathbb{Z})$. Show that I is an ideal of $\mathbb{Z}[x]$. Is it a prime ideal? Is it a maximal ideal? Give proper justification in support of your answer. 2+2+1=5

(b) Let V be a vector space over the field F and $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ be a basis for V. Let $\beta \in V$ be a non-null vector such that $\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots. + c_n\alpha_n$ for some $c_1, c_2, \ldots\ldots, c_n \in F$ where $c_k \neq 0$ for some $1 \le k \le n$. Then show that $\{\alpha_1, \alpha_2, \ldots.., \alpha_{k-1}, \beta, \alpha_{k+1}, \ldots., \alpha_n\}$ is also a basis for V. 5

(c) Let $(F, +, .)$ be a field and $u(\neq 0) \in F$. Define multiplication × in F by $a \times b = a.u.b$ for $a, b \in F$. Prove that $(F, +, \times)$ is a field.

(d) Find a non-identity linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ such that the $T(W)=W$ where $W=\{(x,y,z)\in \mathbb{R}^3 \mid x+y+z=0\}$.

(e) Let R be a ring and S be a non-empty subset of R. Show that $M=\{a \in R \mid ax=0 \text{ for all } x \in S\}$ is a left ideal of R. Give an example to show that M need not be always an ideal of R. 2+3=5

(f) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(a,b,c)=(a+b, 2c-a)$ for all $(a,b,c)\in \mathbb{R}^3$. Find the matrix representation of T relative to the pair of bases $B=\{(1,0,-1),(1,1,1),(1,0,0)\}$ and $B'=\{(0,1),(1,0)\}$.

3. Answer any ***three*** questions : 10×3=30

(a) (i) Let R be the ring of all continuous function from $\mathbb{R}$ to $\mathbb{R}$. Show that $A=\{f \in R \mid f(0)=0\}$ is a maximal ideal of R.

(ii) Check whether the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$ is isomorphic.

(iii) Let V be a real vector space with three subspaces P, Q, R satisfying $V = P \cup Q \cup R$. Prove that at least one of P, Q, R must be V itself.

3+3+4=10

(b) (i) Let I and J be two ideals of a ring R. Find the smallest ideal of R containing both I and J.

(ii) Give an example to show that quotient ring of an integral domain is not always an integral domain.

(iii) The matrix of a linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $B=\{(-1,1,1),(1,-1,1),(1,1,-1)\}$ is $A=\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 3\\ 3 & 3 & 1\end{pmatrix}$. Find the matrix of T with respect to the ordered basis $B_1=\{(0,1,1),(1,0,1),(1,1,0)\}$.

3+2+5=10

(c) (i) Let $S = \{\frac{a}{b} \in \mathbb{Q} \mid \gcd(a, b) = 1$ and 3 does not divide $b\}$. Show that S is a ring under usual addition and multiplication of rational numbers. Also prove that $M = \{\frac{a}{b} \in S \mid 3$ divides $a\}$ is an ideal of S and the quotient ring S/M is a field.

(ii) Let V, W be two finite dimensional vector spaces over the same field $F, T: V \to W$ be a linear transformation. Then prove that following are equivalent : (A) T carries each linearly independent subset of V to a linearly independent subset of W. (B) $\ker T = \{\theta\}$. (2+2+2)+4=10

(d) (i) Prove that the ring Z_n is s principal ideal ring.

(ii) Find all non-trivial ring homomorphisms from the ring $\mathbb{Z}_{12}$ to the ring $\mathbb{Z}_{28}$.

(iii) Let U, V, W be three finite dimensional vector spaces over the field $F, T: V \to W$ be a linear transformation and $S: W \to U$ be an isomorphism. Then prove that (A) dim ker T = dim ker ST and (B) dim ImT = dim Im ST. 3+3+4=10

(e) (i) In a commutative ring R with unity, then show that an ideal P is a prime ideal if and only if the quotient ring $\frac{R}{P}$ is an integral domain.

(ii) Give an example of an infinite ring with finite characteristic.

(iii) Let U and W be two subspaces of a vector space V over the field F. Prove that $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.

4+2+4=10

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VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : SEC 2 - T

Full Marks : 40

Time : 2 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

[GRAPH THEORY]

Group - A

1. Answer any ***four*** questions : 5×4=20

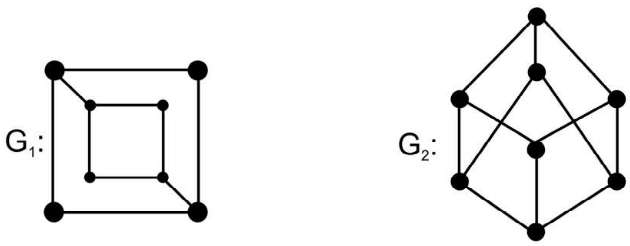
(a) Let G be a graph of order three with the vertex set $V(G)=\{v_1,v_2,v_3\}$. The adjacency matrix is given below :

$A(G)=\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that the graph is disconnected. Draw the graph.

P.T.O.

(b) A connected graph G is an Eulerian graph if and only if every vertex of G has even degree. 5

(c) Define graphs isomorphism. Check whether the following two graphs are isomorphic or not. 2+3



(d) Define Hamiltonian cycle. Draw a graph which is Hamiltonian but not Eulerian. Show that in a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian cycles. 2+1+2

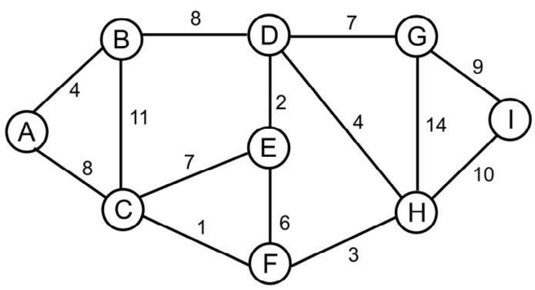
(e) Define a tree. Prove that a tree with n vertices has $n-1$ edges. 1+4

(f) Define spanning tree of a graph G. Show that every connected graph has at least one spanning tree. 1+4

Group - B

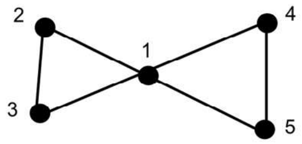
2. Answer any ***two*** questions : 10×2=20

(a) Define weighted shortest path between two vertices. Apply Dijkstra's algorithm to the graph given below and find the shortest path from the vertex 0 to the vertex 4. 2+8



(b) Define a weighted graph. Describe Warshall algorithm to find all-pairs shortest paths. 2+8

(c) Define the root of a rooted tree. Prove that there is one and only one path between every pair of vertices in a tree. Draw all spanning trees from the following graph. 1+3+6



(d) Define an Eulerian graph. Write a short note on travelling salesman's problem. Prove that a simple (having no self-loops and parallel edges) graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges. 1+3+6

OR

[COMPUTER GRAPHICS]

1. Answer any ***four*** questions : 5×4=20

   (a) Discuss raster scan approach.

   (b) Explain the concept of Pixel, Aspect Ratio, and Resolution.

   (c) Describe CMYK Color Model.

   (d) Briefly discuss the Flood Fill algorithm.

   (e) What is meant by Anti-Aliasing?

   (f) Define convex and concave polygon.

2. Answer any ***two*** questions : 10×2=20

   (a) Consider the line from (0, 0) to (4, 6). Use DDA algorithm to rasterize this line.

   (b) Discuss Midpoint Circle Drawing algorithm.

   (c) Explain 2D transformations with its basic types.

   (d) Write algorithm to clip line using Cohen Sutherland line clipping algorithm.

OR

[OPERATING SYSTEM : LINUX]

1. Answer any ***four*** questions : 5×4=20

(a) (i) What is a partition table?

(ii) Compare multitasking and multiuser OS. 2+3

(b) Discuss kernel approach OS structure. 5

(c) Write short note of CPU scheduler. 5

(d) (i) What is scheduling context of process management?

(ii) State the task of fork () and exec () comment? 3+2

(e) Discuss general characteristics of the Ext3 file system. 5

(f) (i) What are the three main purposes of an OS?

(ii) UNIX is multitasking operating system. Why? 3+2

2. Answer any ***two*** questions : 10×2=20

(a) (i) Explain demand paging.

(ii) There is no external fragmentation in paging. Why?

(iii) Compare paging and segmentation scheme. 4+2+4

(b) (i) What is a virtual memory?

(ii) Explain Belady's anomaly with example.

(iii) What is the functionality of "pipes" in shell? 2+6+2

(c) (i) What is cooperating process?

(ii) Compare shared memory system and message passing system in process communication model.

(iii) Compare process and thread.

(iv) What is the difference between virtual address space and physical address space? 2+4+2+2

(d) (i) Why TLB uses in paging memory management scheme?

(ii) Discuss the basic method of paging.

(iii) When paging also suffers from internal fragmentation? 2+5+3